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PLANE CURVES IN AN IMMERSED GRAPH IN $\mathbb{R}^2$

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Abstract. For any chord diagram on a circle there exists a complete graph on sufficiently many vertices such that any generic immersion of it to the plane contains a plane closed curve whose chord diagram contains the given chord diagram as a sub-chord diagram. For any generic immersion of the complete graph on six vertices to the plane the sum of averaged invariants of all Hamiltonian plane curves in it is congruent to one quarter modulo one half.

1. Introduction

Throughout this paper we work in the piecewise linear category. By K_n we denote the complete graph on n vertices. It is shown in [2] and [16] that every embedding of K_6 into the 3-dimensional Euclidean space $\mathbb{R}^3$ always contains a non-splittable 2-component link. It is also shown in [2] that every embedding of K_7 into $\mathbb{R}^3$ always contains a nontrivial knot. There are a number of subsequent results. See for example [10], [8], [15], [3], [5], [4], [13], [11] and [12]. These results show if an abstract graph G is sufficiently complicated then any embedding of G into $\mathbb{R}^3$ contains a complicated knot, link or knotted subgraph with prescribed properties.

In this paper we show some analogous results in two-dimensions. Namely we consider a generic immersion of a finite graph into the Euclidean plane $\mathbb{R}^2$ and consider the immersed plane closed curves contained in the immersed graph.

Let G be a finite graph. We consider G as a topological space in the usual way. A *cycle* is a subgraph of G that is homeomorphic to a circle. An n-cycle of G is a cycle of G that contains exactly n vertices. We denote the set of all n-cycles of G by $\Gamma_n(G)$ and the set of all cycles of G by $\Gamma(G)$. Namely $\Gamma(G) = \cup_{n\in\mathbb{N}}\Gamma_n(G)$. By a *generic immersion* we mean a continuous map from G to $\mathbb{R}^2$ whose multiple points are only finitely many transversal double points of interior points of edges. These double points are called *crossing points* and the number of crossing points of a generic immersion $f : G \to \mathbb{R}^2$ is denoted by $c(f)$. We consider a circle as a graph. A *plane curve* is a generic immersion of a circle. An n-chord diagram is a paired $2n$ points on an oriented circle. An n-chord diagram is also called a *chord diagram*. Two n-chord diagrams $\mathcal{C}_1$ on an oriented circle S_1 and $\mathcal{C}_2$ on an oriented circle S_2 are equivalent if there is an orientation preserving homeomorphism from S_1 to S_2 that maps each pair of points of $\mathcal{C}_1$ to a pair of points of $\mathcal{C}_2$. We consider chord diagrams up to this equivalence relation unless otherwise stated. A *sub-chord diagram* of a chord diagram $\mathcal{C}$ on an oriented circle S is a chord diagram $\mathcal{D}$ on S whose paired points are paired points of $\mathcal{C}$.

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We describe an n-chord diagram by a circle with n-dotted chords. Each chord joins paired points of the chord diagram. See for example Figure 3.1.

Let S be an oriented circle and $f : S \to \mathbb{R}^2$ a plane curve. By $\mathcal{C}(f)$ we denote the chord diagram on S whose paired points correspond to the multiple points of f. Thus $\mathcal{C}(f)$ is a $c(f)$-chord diagram on S. Let G be a finite graph and $f : G \to \mathbb{R}^2$ a generic immersion. Then for each cycle γ of G we have a restriction map $f|_\gamma : \gamma \to \mathbb{R}^2$. Then we have a chord diagram $\mathcal{C}(f|_\gamma)$ on the circle γ. By $\mathbb{S}^1$ we denote the unit circle with counterclockwise orientation.

Theorem 1.1. *Let n be a natural number with $n \geq 2$. Let $\mathcal{C}$ be an n-chord diagram on $\mathbb{S}^1$. Let $f : K_{4n} \to \mathbb{R}^2$ be a generic immersion. Then there is a cycle $\gamma \in \Gamma_{4n}(K_{4n})$ and a sub-chord diagram $\mathcal{D}$ of $\mathcal{C}(f|_\gamma)$ such that $\mathcal{D}$ is equivalent to $\mathcal{C}$.*

An application of Theorem 1.1 to knot projections is shown in Section 3.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the natural projection defined by $\pi(x, y, z) = (x, y)$. A *knot* is a subspace of $\mathbb{R}^3$ that is homeomorphic to a circle. A knot K in $\mathbb{R}^3$ is said to be *in regular position* if the restriction map $\pi|_K : K \to \mathbb{R}^2$ is a generic immersion from the circle K to $\mathbb{R}^2$.

Let S be a circle and $f : S \to \mathbb{R}^2$ a generic immersion. Let $a_2(f)$ be the averaged invariant of the second coefficient of Conway polynomial of knots [14] [9]. Namely $a_2(f)$ is the average of $a_2(K)$ where K varies over all $2^{c(f)}$ knots in $\mathbb{R}^3$ in regular position with $\pi(K) = f(S)$. Here $a_2(K)$ denotes the second coefficient of the Conway polynomial of a knot K. Let $f : K_6 \to \mathbb{R}^2$ be a generic immersion. We define $\alpha(f)$ by

$$\alpha(f) = \sum_{\gamma \in \Gamma_6(K_6)} a_2(f|_\gamma).$$

Theorem 1.2. *Let $f : K_6 \to \mathbb{R}^2$ be a generic immersion. Then*

$$\alpha(f) \equiv \frac{1}{4} \pmod{\frac{1}{2}}.$$

Corollary 1.3. *Let $f : K_6 \to \mathbb{R}^2$ be a generic immersion. Then there is a cycle $\gamma \in \Gamma_6(K_6)$ and a knot K in $\mathbb{R}^3$ in regular position with $a_2(K) \neq 0$ such that $f(\gamma) = \pi(K)$.*

Note that Corollary 1.3 implies the result [6, Theorem 3.4] that every generic immersion of K_6 contains a projection of a nontrivial knot.

Example 1.4. Let f_1, f_2 and f_3 be generic immersions from K_6 to $\mathbb{R}^2$ illustrated in Figure 1.1. Then by a straightforward calculation we have $\alpha(f_1) = \frac{1}{4}, \alpha(f_2) = \frac{3}{4}$ and $\alpha(f_3) = \frac{5}{4}$.

2. Proofs

We denote the set of all vertices of a graph G by $V(G)$ and the set of all edges of G by $E(G)$. Let $K_{m,n}$ be the complete bipartite graph on $m+n$ vertices partitioned into m vertices and n vertices. Let G be the complete graph K_5 or the complete bipartite graph $K_{3,3}$. Let $f : G \to \mathbb{R}^2$ be a generic immersion. Let x and y be mutually disjoint edges of G. We denote the number of crossing points of f

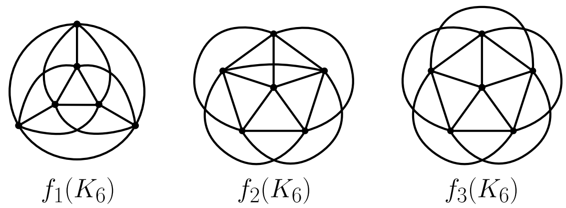

FIGURE 1.1.

contained in $f(x)\cap f(y)$ by $\#(f(x)\cap f(y))$. We denote the set of all unordered pairs of mutually disjoint edges of G by $\mathcal{E}(G)$. Set

$$d(f) = \sum_{(x,y)\in\mathcal{E}(G)} \#(f(x)\cap f(y)).$$

Namely $d(f)$ is the number of crossing points of f made of disjoint edges of G.

Proposition 2.1. *Let G be the complete graph K_5 or the complete bipartite graph $K_{3,3}$. Let $f : G \to \mathbb{R}^2$ be a generic immersion. Then*

$$d(f) \equiv 1 \pmod 2.$$

Proof. Let $f_0 : G \to \mathbb{R}^2$ be a generic immersion illustrated in Figure 2.1. Then we have

$$d(f_0) = 1.$$

It is well-known that any two generic immersions are transformed into each other up to self-homeomorphisms of G and $\mathbb{R}^2$ by a finite sequence of the local moves illustrated in Figure 2.2. It is easy to check that these moves do not change the parity of d if $G = K_5$ or $G = K_{3,3}$. Thus we have the result. $\square$

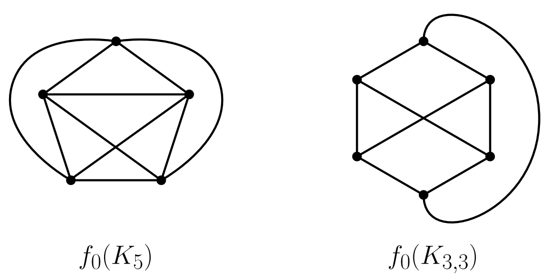

FIGURE 2.1.

Let G be a graph and W a subset of $V(G)$. The induced subgraph $G[W]$ is the maximal subgraph of G with $V(G[W]) = W$.

Proof of Theorem 1.1. We will repeatedly apply Proposition 2.1 for certain subgraphs of K_{4n} each of which is isomorphic to K_5 as follows. Set $V(K_{4n}) = \{v_1, v_2, \cdots, v_{4n}\}$ and $H_1 = K_{4n}[\{v_1, v_2, v_3, v_4, v_5\}]$. Then H_1 is isomorphic to K_5. By Proposition 2.1 there exist disjoint edges x and y of H_1 such that $f(x)\cap f(y) \neq \emptyset$. Let c_1 be a crossing in $f(x)\cap f(y)$. We may suppose without loss of generality that

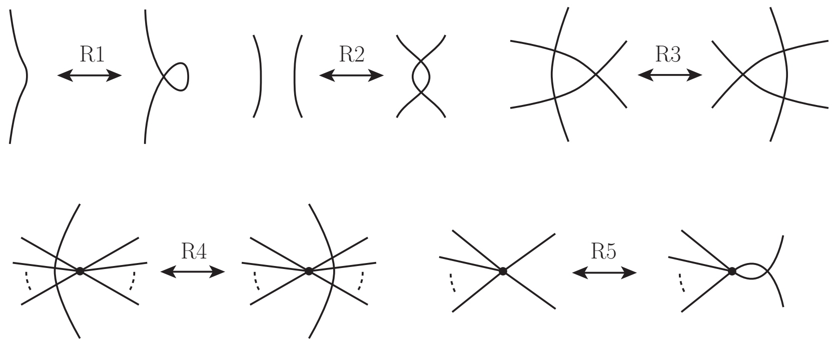

Figure 2.2.

$x = v_1v_2$ and $y = v_3v_4$. Then set $H_2 = K_{4n}[\{v_5, v_6, v_7, v_8, v_9\}]$. By the same argument we may suppose without loss of generality that $f(v_5v_6) \cap f(v_7v_8) \neq \emptyset$. Let c_2 be a crossing in $f(v_5v_6) \cap f(v_7v_8)$. Then set $H_3 = K_{4n}[\{v_9, v_{10}, v_{11}, v_{12}, v_{13}\}]$ and repeat the arguments. Thus we have $f(v_{4(k-1)+1}v_{4(k-1)+2}) \cap f(v_{4(k-1)+3}v_{4(k-1)+4}) \neq \emptyset$ for each $k \in \{1, 2, \cdots, n-1\}$. Let c_k be a crossing in $f(v_{4(k-1)+1}v_{4(k-1)+2}) \cap f(v_{4(k-1)+3}v_{4(k-1)+4})$ for each $k \in \{1, 2, \cdots, n-1\}$. Finally set the subgraph $H_n = K_{4n}[\{v_{4n-4}, v_{4n-3}, v_{4n-2}, v_{4n-1}, v_{4n}\}]$. There are two cases. Namely we may suppose without loss of generality that either $f(v_{4n-3}v_{4n-2}) \cap f(v_{4n-1}v_{4n}) \neq \emptyset$ or $f(v_{4n-4}v_{4n-3}) \cap f(v_{4n-2}v_{4n-1}) \neq \emptyset$. Let c_n be a crossing in $f(v_{4n-3}v_{4n-2}) \cap f(v_{4n-1}v_{4n})$ or $f(v_{4n-4}v_{4n-3}) \cap f(v_{4n-2}v_{4n-1})$ respectively. In either case it is easy to find $\gamma \in \Gamma_{4n}(K_{4n})$ containing the edges $v_1v_2, v_3v_4, \cdots, v_{4n-7}v_{4n-6}, v_{4n-5}v_{4n-4}$ and $v_{4n-3}v_{4n-2}, v_{4n-1}v_{4n}$ or $v_{4n-4}v_{4n-3}, v_{4n-2}v_{4n-1}$ such that the sub-chord diagram $\mathcal{D}$ of $\mathcal{C}(f|_\gamma)$ corresponding to the crossings $c_1, \cdots, c_n$ is equivalent to $\mathcal{C}$. This completes the proof. □

Proposition 2.2. *Let $f_1, g_1, f_{2,1}, g_{2,1}, f_{2,2}, g_{2,2}, f_{3,1}, g_{3,1}, f_{3,2}$ and $g_{3,2}$ be generic immersions from $\mathbb{S}^1$ to $\mathbb{R}^2$ such that each of f_1 and g_1, $f_{2,1}$ and $g_{2,1}$, $f_{2,2}$ and $g_{2,2}$, $f_{3,1}$ and $g_{3,1}$ and $f_{3,2}$ and $g_{3,2}$ differ locally as illustrated in Figure 2.3. Then we have the following formulas.*

(1) $a_2(f_1) - a_2(g_1) = 0$,
(2) $a_2(f_{2,1}) - a_2(g_{2,1}) = 0$,
(3) $a_2(f_{2,2}) - a_2(g_{2,2}) = \frac{1}{4}$,
(4) $a_2(f_{3,1}) - a_2(g_{3,1}) = \frac{1}{4}$,
(5) $a_2(f_{3,2}) - a_2(g_{3,2}) = \frac{1}{4}$.

Proof. It is known in [14] that $a_2 = \frac{1}{8}(J^+ + 2St)$ where J^+ and St are plane curve invariants defined by Arnold [1]. Then the formulas follow from the definitions of these invariants. We note that a direct calculation based on the well known formula $a_2(K_+) - a_2(K_-) = \mathrm{lk}(L_0)$ in [7] where K_+, K_- and L_0 are knots and a 2-component link forming a skein triple and lk denotes the linking number is also straightforward. □

Proof of Theorem 1.2. Let $f_1 : K_6 \to \mathbb{R}^2$ be a generic immersion in Example 1.4. Then $\alpha(f_1) = \frac{1}{4}$. Let $f : K_6 \to \mathbb{R}^2$ be any generic immersion. Then f and f_1 are transformed into each other up to self-homeomorphisms of G and $\mathbb{R}^2$ by

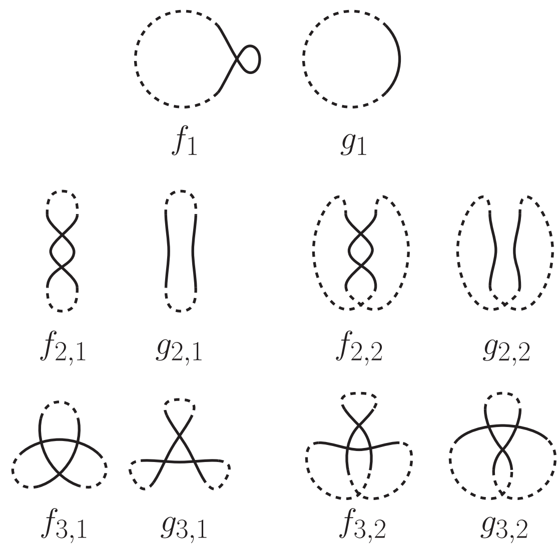

FIGURE 2.3.

a finite sequence of the local moves $\mathrm{R1}, \mathrm{R2}, \mathrm{R3}, \mathrm{R4}, \mathrm{R5}$ illustrated in Figure 2.2. Therefore it is sufficient to show that the parity of 4α is invariant under these five local moves. Let $f : K_6 \to \mathbb{R}^2$ and $g : K_6 \to \mathbb{R}^2$ be generic immersions that differ by an application of a local move R where R is one of $\mathrm{R1}, \mathrm{R2}, \mathrm{R3}, \mathrm{R4}$ and R5. We consider the following four cases.

Case 1. $\mathrm{R} = \mathrm{R1}$ or $\mathrm{R} = \mathrm{R5}$. It follows from Proposition 2.2 (1) that $a_2(f|_\gamma)$ is invariant under R1 and R5 for each 6-cycle $\gamma \in \Gamma_6(K_6)$. Therefore $\alpha(f)$ is invariant under R1 and R5.

Case 2. $\mathrm{R} = \mathrm{R2}$. Let e_1 and e_2 be the edges of K_6 involved with R2.

Case 2.1. $e_1 = e_2$. In this case the 4! 6-cycles of K_6 containing e_1 are involved. Note that all of them differ as $f_{2,1}$ and $g_{2,1}$ in Figure 2.3, or all of them differ as $f_{2,2}$ and $g_{2,2}$ in Figure 2.3. In the former case we apply Proposition 2.2 (2) and we have $\alpha(f) = \alpha(g)$. In the latter case we apply Proposition 2.2 (3) and we have $\alpha(f) - \alpha(g) = \pm\frac{4!}{4}$. Since 4! is even the parity of $4\alpha(f)$ and $4\alpha(g)$ coincide.

Case 2.2. e_1 and e_2 are adjacent. In this case the 3! 6-cycles containing both e_1 and e_2 are involved. Since 3! is even we have the conclusion by the same arguments as Case 2.1.

Case 2.3. e_1 and e_2 are disjoint. In this case the 12 6-cycles containing both e_1 and e_2 are involved. Exactly half of them are as $f_{2,1}$ and $g_{2,1}$ in Figure 2.3, and the other half of them are as $f_{2,2}$ and $g_{2,2}$ in Figure 2.3. Since the one half of 12 is even, we have the conclusion.

Case 3. $\mathrm{R} = \mathrm{R3}$. Let e_1, e_2 and e_3 be the edges of K_6 involved with R3.

Case 3.1. $e_1 = e_2 = e_3$. In this case the 4! 6-cycles of K_6 containing e_1 are involved. Note that all of them differ as $f_{3,1}$ and $g_{3,1}$ in Figure 2.3, or all of them differ as $f_{3,2}$ and $g_{3,2}$ in Figure 2.3. In the former case we apply Proposition 2.2 (4) and we have $\alpha(f) = \alpha(g) = \pm\frac{4!}{4}$. In the latter case we apply Proposition 2.2 (5) and we have $\alpha(f) - \alpha(g) = \pm\frac{4!}{4}$. Since 4! is even the parity of $4\alpha(f)$ and $4\alpha(g)$ coincide.

Case 3.2. Exactly two of e_1, e_2 and e_3 are the same edge. We may suppose without loss of generality that $e_1 = e_2$ and $e_3 \neq e_1$. First suppose that e_1 and e_3 are adjacent. Then there are 3! 6-cycles of K_6 containing e_1 and e_3 and by applying Proposition 2.2 (4) or (5), we have the conclusion. Next suppose that e_1 and e_3 are disjoint. Then there are 12 6-cycles of K_6 containing both of e_1 and e_3. Exactly half of them are as $f_{3,1}$ and $g_{3,1}$ in Figure 2.3, and the other half of them are as $f_{3,2}$ and $g_{3,2}$ in Figure 2.3. Thus we have the conclusion.

Case 3.3. $e_1 \neq e_2 \neq e_3 \neq e_1$. We consider the following four cases.

Case 3.3.1. $e_1 \cup e_2 \cup e_3$ is homeomorphic to a circle. In this case no 6-cycle of K_6 are involved and we have the result.

Case 3.3.2. $e_1 \cap e_2 \cap e_3 \neq \emptyset$. In this case no 6-cycle of K_6 are involved and we have the result.

Case 3.3.3. $e_1 \cup e_2 \cup e_3$ is homeomorphic to a closed interval. In this case the 2 6-cycles containing all of e_1, e_2 and e_3 are involved and both of them are as $f_{3,1}$ and $g_{3,1}$ in Figure 2.3, or both of them are as $f_{3,2}$ and $g_{3,2}$ in Figure 2.3. Thus we have the conclusion.

Case 3.3.4. e_1 and e_2 are adjacent and e_3 is disjoint from $e_1 \cup e_2$. In this case the 4 6-cycles containing all of e_1, e_2 and e_3 are involved and exactly two of them are as $f_{3,1}$ and $g_{3,1}$ in Figure 2.3, and the other two of them are as $f_{3,2}$ and $g_{3,2}$ in Figure 2.3. Thus we have the conclusion.

Case 3.3.5. No two of e_1, e_2 and e_3 are adjacent. In this case the 8 6-cycles containing all of e_1, e_2 and e_3 are involved and exactly four of them are as $f_{3,1}$ and $g_{3,1}$ in Figure 2.3, and the other four of them are as $f_{3,2}$ and $g_{3,2}$ in Figure 2.3. Thus we have the conclusion.

Case 4. R = R4. Up to local move R2, we may suppose without loss of generality that the local move R4 on K_6 is as illustrated in Figure 2.4. Note that the edges e_1, e_2, e_3, e_4 and e_5 in Figure 2.4 are mutually distinct and e_6 may be equal to one of these five edges. For each pair $1 \leq i < j \leq 5$, let $\Gamma_{i,j}$ be the set of 6-cycles of K_6 that contains all of the edges e_i, e_j and e_6. Since

$$\alpha(f) - \alpha(g) = \sum_{\gamma \in \Gamma_6(K_6)} (a_2(f|_\gamma) - a_2(g|_\gamma))$$

and $a_2(f|_\gamma) = a_2(g|_\gamma)$ for each $\gamma \in \Gamma_6(K_6) \setminus \bigcup_{1 \leq i < j \leq 5} \Gamma_{i,j}$, it is sufficient to show that

$$\sum_{\gamma \in \Gamma_{i,j}} 4(a_2(f|_\gamma) - a_2(g|_\gamma))$$

is even for each pair $1 \leq i < j \leq 5$.

Case 4.1. $e_6 = e_i$ or $e_6 = e_j$. The 3! 6-cycles containing both of e_i and e_j are involved. By applying Proposition 2.2 (2) or (3), we have the conclusion.

Case 4.2. $e_6 \neq e_i$ and $e_6 \neq e_j$. If $e_i \cup e_j \cup e_6$ is homeomorphic to a circle or the complete bipartite graph $K_{1,3}$, then no 6-cycle of K_6 is involved. Suppose that $e_i \cup e_j \cup e_6$ is homeomorphic to a closed interval. Then the two 6-cycles of K_6 containing all of e_i, e_j and e_6 are involved. By applying Proposition 2.2 (2) or (3), we have the conclusion. Suppose that e_6 is disjoint from $e_i \cup e_j$. Then the 4 6-cycles containing all of e_i, e_j and e_6 are involved. Exactly two of them are as $f_{2,1}$ and $g_{2,1}$ in Figure 2.3, and the other two of them are as $f_{2,2}$ and $g_{2,2}$ in Figure 2.3. Thus we have the conclusion. $\square$

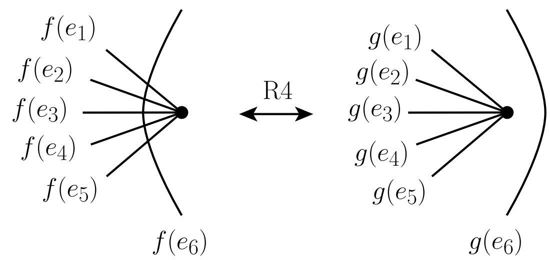

FIGURE 2.4.

Remark 2.3. Let $K_{l,m,n}$ be the complete tripartite graph on $l+m+n$ vertices partitioned into l vertices, m vertices and n vertices. It is known that as well as K_6, every embedding of the complete tripartite graph $K_{3,3,1}$ into the 3-dimensional space $\mathbb{R}^3$ contains a non-splittable 2-component link [16]. Here we show two generic immersions of $K_{3,3,1}$ to $\mathbb{R}^2$ that suggest non-existence of an analogy of Theorem 1.2 for $K_{3,3,1}$. Let $f : K_{3,3,1} \to \mathbb{R}^2$ be a generic immersion. We define $\alpha(f)$ by the following.

$$\alpha(f) = \sum_{\gamma \in \Gamma_7(K_{3,3,1})} a_2(f|_\gamma).$$

Let f_1 and f_2 be generic immersions of $K_{3,3,1}$ to $\mathbb{R}^2$ illustrated in Figure 2.5. Then by a straightforward calculation we have $\alpha(f_1) = \frac{3}{4}$ and $\alpha(f_2) = 1$.

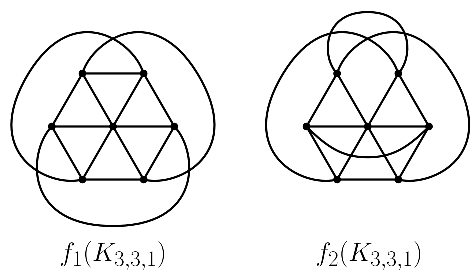

FIGURE 2.5.

3. KNOT PROJECTIONS IN A PLANE IMMERSED GRAPH

In this section we show an application of Theorem 1.1 to knot projections. Two knots K_1 and K_2 are said to be *ambient isotopic* if there is an orientation preserving self-homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K_1) = K_2$. Let K be a knot in $\mathbb{R}^3$. Let $f : \mathbb{S}^1 \to \mathbb{R}^2$ a generic immersion. We say that f is a *regular projection of* K if there exists a knot K' in regular position that is ambient isotopic to K such that $\pi(K') = f(\mathbb{S}^1)$. Let $\mathbb{R}^2 \cup \infty$ be the one-point compactification of $\mathbb{R}^2 \cup \infty$ and $\iota : \mathbb{R}^2 \to \mathbb{R}^2 \cup \infty$ the natural inclusion map. Two plane curves $f : \mathbb{S}^1 \to \mathbb{R}^2$ and $g : \mathbb{S}^1 \to \mathbb{R}^2$ are said to be *spherically equivalent* if there exists an orientation preserving homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ and an orientation preserving homeomorphism $h : \mathbb{R}^2 \cup \infty \to \mathbb{R}^2 \cup \infty$ such that $h \circ \iota \circ f = \iota \circ g \circ \varphi$. Let $f : \mathbb{S}^1 \to \mathbb{R}^2$ be a plane curve and C a circle in $\mathbb{R}^2$ transversely intersecting $f(\mathbb{S}^1)$ at two points, say P_1 and P_2.

Let A be a simple arc in C joining P_1 and P_2. Let $p : \mathbb{S}^1 \to \mathbb{R}^2$ and $q : \mathbb{S}^1 \to \mathbb{R}^2$ be plane curves such that $p(\mathbb{S}^1) \cup q(\mathbb{S}^1) = f(\mathbb{S}^1) \cup A$, $p(\mathbb{S}^1) \cap q(\mathbb{S}^1) = A$, the orientation of p coincides with that of f on $p(\mathbb{S}^1) \setminus A$ and the orientation of q coincides with that of f on $q(\mathbb{S}^1) \setminus A$. Then we say that f is a *connected sum of p and q*. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be chord diagrams illustrated in Figure 3.1. The following Theorem 3.1 and Theorem 3.2 are implicitly shown in the proofs of Theorem 1 and Theorem 2 of [17] respectively.

Theorem 3.1. *Let $f : \mathbb{S}^1 \to \mathbb{R}^2$ be a generic immersion. Then the following conditions are mutually equivalent.*

(1) f is a regular projection of some nontrivial knot K.

(2) f is a regular projection of the trefoil knot.

(3) $f(\mathbb{S}^1)$ is not obtained from an embedded circle in $\mathbb{R}^2$ by repeated applications of the local move R_1 illustrated in Figure 2.2.

(4) The chord diagram $\mathcal{C}(f)$ contains a sub-chord diagram that is equivalent to $\mathcal{C}_1$.

Theorem 3.2. *Let $f : \mathbb{S}^1 \to \mathbb{R}^2$ be a generic immersion. Then the following conditions are mutually equivalent.*

(1) f is a regular projection of the figure-eight knot.

(2) $f(\mathbb{S}^1)$ is not equivalent to any connected sum of some plane curves each of which is equivalent to one of the plane curves $U, T, P_1, P_2, P_3, \cdots$ illustrated in Figure 3.2.

(3) The chord diagram $\mathcal{C}(f)$ contains a sub-chord diagram that is equivalent to $\mathcal{C}_2$.

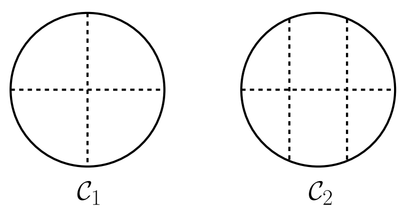

Figure 3.1.

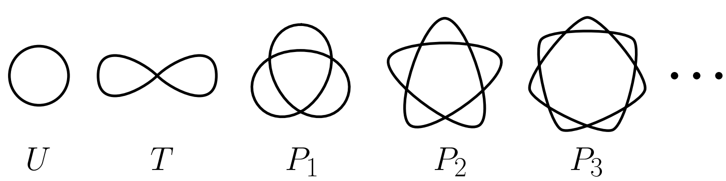

Figure 3.2.

Note that Theorem 3.1 and [6, Theorem 3.4] implies that for any generic immersion $f : K_6 \to \mathbb{R}^2$ there exists a 6-cycle γ of K_6 such that $f(\gamma)$ is a regular projection of the trefoil knot. By Theorem 1.1 and Theorem 3.2 we have the following corollary.

Corollary 3.3. *Let $f : K_{12} \to \mathbb{R}^2$ be a generic immersion. Then there is a cycle $\gamma \in \Gamma_{12}(K_{12})$ such that $f(\gamma)$ is a regular projection of the figure-eight knot.*

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